Edge-Based BE-FE Coupling for Electromagnetics

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Abstract—Electromagnetic problems can be solved using the coupled BE-FE method, where the conducting and magnetic parts are discretized by finite elements and the surrounding space is described with the help of the boundary element method. In this paper we propose a novel BE-FE coupling procedure for the solution of electromagnetic problems. Starting from the potential formulation we derive a representation formula in the BEM domain in terms of differential forms. The corresponding boundary integral equation will be discretized by the DeRham map, generalizing the classical point collocation [12]. The Cauchy data will be discretized by the first-order Whitney elements [13] on the boundary. In the FE domain we apply the Galerkin discretization scheme using the spacial first-order Whitney elements. The coupling of these methods will be described and the feasibility of the proposed method will be demonstrated by means of numerical examples.

I. INTRODUCTION

For coupling purposes, the computational domain $\mathbb{R}^3$ is decomposed into two subdomains, $\mathbb{R}^3 = \Omega^+ \cup \Gamma \cup \Omega^-$. The bounded subdomain $\Omega^-$ can contain conducting and permeable materials, whereas $\Omega^+$ represents the surrounding air. The common boundary $\Gamma$ of both subdomains is supposed to be piecewise smooth. In $\Omega^-$ the FEM will be applied, while in $\Omega^+$ we derive the representation formula for the magnetic vector potential. The discretization of the corresponding boundary integral equation will be carried out by a novel collocation procedure via DeRham maps [12] which is an alternative approach to the Galerkin-BEM [9] and can be seen as a generalization of the classical point collocation.

II. BOUNDARY INTEGRAL EQUATION

It is well known that electromagnetic fields can be treated as differential forms of certain degrees [5], [6], [7]. A differential form of degree $p$ ($p$-form) $\omega$ on a domain $\Omega$ can be integrated over compact oriented $p$-dimensional submanifolds (also called $p$-chains) of $\Omega$.

$$\omega : C_p(\Omega) \mapsto \mathbb{R}.$$

This mapping yields in particular a natural evaluation of $p$-forms on $p$-dimensional objects. In the following we denote by $\mathcal{F}^p(\Omega)$ the space of $p$-forms on $\Omega$. The Maxwell’s equations can be written in terms of differential forms

$$dE = -\partial_t B, \quad dH = J + \partial_t D,$$

where $E, H \in \mathcal{F}^1(\mathbb{R}^3)$, $B, D, j \in \mathcal{F}^2(\mathbb{R}^3)$, and differential operators has been generalized by the exterior derivative

$$d : \mathcal{F}^p(\mathbb{R}^3) \mapsto \mathcal{F}^{p+1}(\mathbb{R}^3).$$

The material laws

$$\mu H = \ast B, \quad \varepsilon E = \ast D$$

couple $p$- and $(3-p)$-forms and are represented by Hodge-operators

$$\ast : \mathcal{F}^p(\mathbb{R}^3) \mapsto \mathcal{F}^{3-p}(\mathbb{R}^3).$$

Starting from the quasistatic Ampère’s law

$$dH = J,$$

using the potential ansatz $B = dA, E = -\partial_t A - d\varphi$ with $A \in \mathcal{F}^1(\mathbb{R}^3), \varphi \in \mathcal{F}^0(\mathbb{R}^3)$, and the material law $\mu H = \ast B$ we obtain the second order differential equation

$$d \ast \mu dA = J. \quad (1)$$

A necessary condition for the solvability of (1) is $dJ = 0$. Furthermore, we require the vanishing trace of $J$ on $\Gamma$

$$t_J = 0, \quad (2)$$

to assure that no currents enter the subdomain $\Omega^+$. For the derivation of the representation formula we use the following dyadic Green’s function

$$U^*(x, y) = \frac{I}{4\pi|x-y|},$$

with the unit double form

$$I = dx_1 \otimes dy_1 + dx_2 \otimes dy_2 + dx_3 \otimes dy_3.$$

$U^*$ is the fundamental solution of the Laplace operator $\Delta = (-\ast d \ast d + d \ast d \ast)$. Pre-multiplying of (1) by $U^*$, integrating over $\Omega^+$ and performing integration by parts yields the following representation formula

$$\Delta = -\int_{\Omega^+} (\gamma^+ \ast U^*) \ast (\gamma^+ \Delta) + \int_{\Omega^+} (\gamma^+ \ast U^*) \ast (\gamma^+ \partial^+ \Delta)$$

$$+ \int_{\Omega^+} U^* \ast \partial^+ \ast \int_{\Omega^+} \gamma_+ \ast U^* \ast \gamma_+ \partial^+ \Delta \quad (3)$$
with trace operators \( \gamma_D = t \cdot \), \( \gamma_N = t \times d \cdot \), \( \gamma_D = t \times d \cdot \) and \( \gamma_N = t \times d \cdot \). We denote by \( \beta \) and \( \gamma \) the Dirichlet data \( \gamma_D^+ \Delta \) and the Neumann data \( \gamma_N^+ \Delta \), respectively, and rewrite (3) in terms of potentials

\[ \Delta = \Psi_{SL} \beta + \Psi_{DL} \gamma + \Psi_{Newton}(\frac{1}{2}) + \Psi_{Gauge}(\gamma_D^+ \Delta). \]

The layer potentials \( \Psi_{SL} \) and \( \Psi_{DL} \) satisfy the following jump relations on \( \Gamma \)

\[ [\gamma_D \Psi_{SL}] = 0, \quad [\gamma_D \Psi_{DL}] = -\mathcal{I}, \]

where \( \mathcal{I} \) is the identity and \([\cdot]\) = \( \gamma_D^+ \cdot - \gamma_D^- \) is the jump across \( \Gamma \). We define boundary integral operators

\[ \mathcal{V} = \gamma_D^+ \circ \Psi_{SL}, \quad \mathcal{G} = \gamma_D^+ \circ \Psi_{Gauge}, \]

\[ \mathcal{K} = \left( \frac{\Theta^+ \gamma_D^+}{4\pi} + \frac{\Theta^- \gamma_D^-}{4\pi} \right) \circ \Psi_{DL}, \]

where \( \Theta^+ \) and \( \Theta^- \) are the exterior and interior solid angles. By applying the Dirichlet trace operator \( \gamma_D^+ \) to the representation formula and taking into account the jump relations of the layer potentials we obtain the boundary integral equation

\[ \mathcal{V} \gamma + \mathcal{G}(\gamma_D^+ \Delta) = \left( \frac{\Theta^+}{4\pi} \mathcal{I} + \mathcal{K} \right) \beta + \beta^{\text{dir}}. \tag{4} \]

From \( d\mathcal{G} = 0 \) and the Stokes theorem

\[ \omega |_{\partial E} = d\omega |_{E} \]

we obtain that the gauge potential vanish by evaluation on closed chains (cycles)

\[ \mathcal{G}(\gamma_D^+ \Delta) |_{\partial E} = 0 \quad \text{for all cycles} \quad \partial E \in \mathcal{C}_1(\partial_0, \Gamma). \tag{5} \]

We define the DeRham map as a bilinear form evaluating \( p \)-forms on \( p \)-dimensional oriented submanifolds

\[ P_{\text{DR},p} : \mathcal{F}^p(\Gamma) \times \mathcal{C}_p(\Gamma) \rightarrow \mathbb{R}, \]

\[ (\omega, c) \mapsto \omega |_{c} = \int_c \omega. \]

Similar to the classical point collocation where a scalar boundary integral equation is enforced to be satisfied in some points, the generalized collocation method will enforce the corresponding boundary integral equation to be satisfied on appropriate manifolds. In our special case (4) describes the identity of two \( 1 \)-forms and therefore the application of one-dimensional evaluation by \( P_{\text{DR},1} \) makes sense. This yields the so-called edge collocation method. From (5) we conclude that performing of the edge collocation on cycles instead of general \( 1 \)-chains eliminates the contribution of the gauge potential. We obtain the collocation formulation for the Dirichlet-type problem

\[ \text{For a given} \ \beta \in \mathcal{F}^0(\Gamma) \text{ find} \ \gamma \in \mathcal{F}^1(\partial_0, \Gamma) \text{ such that} \]

\[ P_{\text{DR},1}(V \gamma, \partial_0) = P_{\text{DR},1}(\left( \frac{\Theta^+}{4\pi} \mathcal{I} + \mathcal{K} \right) \beta + \beta^{\text{dir}}, \partial_0) \tag{6} \]

holds for all cycles \( \partial_0 \in \mathcal{C}_1(\partial_0, \Gamma). \)

### III. Discretization

Let \( \Omega^+_h \) be the discretization of the inner domain \( \Omega^- \) and \( \Gamma_h \) the corresponding discretization of the boundary \( \Gamma \). Ampère’s and Faraday’s laws might be discretized on dual grid systems, and the coupling of these is accomplished by material matrices which can be seen as discrete Hodge operators [8]. Application of Hodge operators is associated with a change of the grid on the discrete level. Therefore the Dirichlet and the Neumann data will be allocated on different grids. In the following we denote the primal boundary grid by \( \Gamma_h \) and the dual one by \( \overline{\Gamma}_h \). The unknown data will always be discretized on the primal grid.

In the following this will be the Neumann data since in the BE-FE coupling procedure the exterior Dirichlet-type problem will be solved.

It follows from (2) that the Neumann data are always closed, i.e.

\[ d\gamma = 0 \]

and can therefore according to the DeRham theorem be represented as

\[ \gamma = d\varphi + \Lambda, \quad \varphi \in \mathcal{F}^0(\Gamma), \quad \Lambda \in \mathcal{H}^1(\Gamma), \]

where \( \mathcal{H}^1(\Gamma) \) is a finite-dimensional cohomology space of \( \Gamma \) whose discretization \( \mathcal{H}^1(\Gamma_h) \) is considered in [10]. The discretization of Cauchy data is accomplished by Whitney 1-forms

\[ \beta \approx \beta_h \in \mathcal{W}^1(\overline{\Gamma}_h), \quad \gamma \approx \gamma_h = d\varphi_h + \Lambda_h \in \mathcal{W}^0(\Gamma_h) \oplus \mathcal{H}^1(\Gamma_h). \tag{7} \]

This means that the collocation formulation (6) is now applied to discrete forms. The number of unknowns for the Dirichlet-type problem except the cohomology part is equal to the number of nodes \( n \) in the primal grid \( \Gamma_h \). Therefore the edge collocation (6) should yield \( n \) equations. Choosing the boundaries of dual faces be the collocation paths (Figure 1) we obtain a square single layer matrix and additionally vanishing gauge potential.

Fig. 1: Dirichlet Problem: Edge collocation enforces the boundary integral equation to be satisfied by means of integrals over dual cycles.

By introducing the BEM collocation matrices

\[ [\mathcal{V}_h] = P_{\text{DR},1}(V, C_1(\partial_0, \overline{\Gamma}_h)), \]

\[ [\mathcal{K}_h] = P_{\text{DR},1}(\left( \frac{\Theta^+}{4\pi} \mathcal{I} + \mathcal{K} \right), C_1(\partial_0, \overline{\Gamma}_h)) \]
and the right hand side \( \{ \beta^R_h \} = \mathcal{P}_{DR,1}(\mathcal{P}^{\text{rec}}, C_1(\delta_0, \Gamma_h)) \) the BEM system of equations reads
\[
\{ \gamma_h \} - [K_h]\{ \beta_h \} = \{ \beta^R_h \}.
\] (9)

The matrix \( \gamma_h \) is a non-symmetric square matrix with one-dimensional kernel \( \ker \gamma_h = \text{span}\{1, \ldots, 1\} \), which is easy to regularize. Additionally, the fully populated BEM matrices are generated by asymptotically smooth integral kernels. This fact allows the compression of the matrices by e.g. the ACA method [3], [4] leading to an almost linear complexity of the BEM part.

IV. FE-FORMULATION

We derive a weak formulation of the field problem
\[
d \times \mu \mathbf{A} + \kappa \partial_t \mathbf{A} = 0
\]
with \( \{ \gamma_N \mathbf{A} \} = 0 \). Multiplication with test-forms \( \mathbf{w} \in \mathcal{W}^1(\Omega_h^-) \) and integration over \( \Omega_h^- \) yields
\[
\int_{\Omega_h^-} d \times \mu \mathbf{A} \wedge \mathbf{w} = 0, \quad \forall \mathbf{w} \in \mathcal{W}^1(\Omega_h^-).
\]
Using Stokes’ Theorem
\[
\int_{\Omega_h^-} d \mathbf{F} = \int_{\Gamma_h} \mathbf{F}
\]
and
\[
d(\mathbf{F} \wedge \mathbf{G}) = d\mathbf{F} \wedge \mathbf{G} + (-1)^p \mathbf{F} \wedge d\mathbf{G},
\]
where \( \mathbf{F} \) is a differential form of degree \( p \), we can transform the equation into
\[
\int_{\Omega_h^-} \ast \mu \mathbf{A} \wedge \mathbf{w} - \int_{\Omega_h^-} \gamma_N \mu \mathbf{A} \wedge \gamma_D \mathbf{w} + \int_{\Omega_h^-} \ast \kappa \partial_t \mathbf{A} \wedge \mathbf{w} = 0, \quad \forall \mathbf{w} \in \mathcal{W}^1(\Omega_h^-).
\]
Approximation of \( \mathbf{A} \) by \( \mathbf{A}_h = \sum_{i=1}^{N_e} \mathbf{A}_i \mathbf{w}_i \), \( \mathbf{w}_i \in \mathcal{W}^1(\Omega_h^-) \), yields
\[
\sum_{i=1}^{N_e} \left( \int_{\Omega_h^-} \ast \mu \mathbf{A}_i \mathbf{w}_i \wedge d\mathbf{w}_i - \int_{\Gamma_h} \gamma_N \mu \mathbf{A}_i \mathbf{w}_i \wedge \gamma_D \mathbf{w}_i \right) + \int_{\Omega_h^-} \ast \kappa \partial_t \mathbf{A}_i \mathbf{w}_i \wedge \mathbf{w}_i = 0, \quad \forall \mathbf{w}_i \in \mathcal{W}^1(\Omega_h^-),
\]
where \( N_e \) is the number of edges in the finite element mesh and the \( \mathbf{A}_i \) are the degrees of freedom, i.e., the vector potential circulations along the edges of the mesh.

The equation can be written in matrix form
\[
\{ Q_h \} \{ \Delta_h \} + \{ T_h \} \{ \gamma_h \} = 0,
\]
where the degrees of freedom \( \{ \Delta_h \} \) corresponding to the vector potential will be separated into inner ones \( \{ \alpha_h \} \) and the Dirichlet data \( \{ \beta_h \} \) on the boundary.

V. BE-FE COUPLING

The global system of equations resulting from the BE-FE coupling reads as follows
\[
\begin{pmatrix}
\{ Q_h \} & 0 \\
0 & [\gamma_h] \\
\end{pmatrix}
\begin{pmatrix}
\{ \alpha_h \} \\
\{ \beta_h \} \\
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
\end{pmatrix}
\]

The global system matrix, \( \{ S_h \} \) is singular. The following important properties have been proven in [1]
- The kernel of \( \{ Q_h \} \) is formed of discrete gradients. It has the dimension \( N_n - d \), with \( N_n \) the number of nodes and \( d \) the number of disjoint domains.
- The kernel of \( \{ Q_h \}^T \) is equal to the kernel of \( \{ Q_h \} \).
- \( \{ \gamma_h \} \) has the same kernel as \( \{ Q_h \} \), consisting of constant coefficient vectors.
- The kernel of \( \{ T_h \} \) is equal to the kernel of \( \{ K_h \} \) and is given by constant coefficient vectors; the dimension of the kernel is 1.
- Finally, the kernel of \( \{ K_h \} \) is equal to the kernel of \( \{ T_h \} \) and consists of the discrete gradients on the boundary of the finite element mesh.

As a consequence of all this we can state that, although the BE-FE system matrix is non-symmetric, its kernel is identical to the kernel of its transpose, i.e., the image of the system matrix is orthogonal to its kernel. This fact is important for considering solver strategies.

Due to the explicitly known kernel of the system matrix, we are able to construct a regularization by taking an additional term \( \{ U_h \}[U_h]^T \), where the columns of \( \{ U_h \} \) are basis vectors spanning the kernel. The regularized matrix \( \{ S_h \} + \alpha[\{ U_h \}[U_h]^T \) remains sparse. However, the scaling factor \( \alpha \) has to be chosen properly with respect to the condition number. The regularized matrix can be solved with a preconditioned GMRES solver.

An alternative approach consists in the use of an adapted GMRES solver, [1]. This iterative solver can solve a singular system by an implicit projection of the solution vector into the image of the matrix - provided that kernel and image are orthogonal. This solver avoids the regularization. We are left, however, with the problem of constructing an effective preconditioning for a singular matrix. For this reason the upper approach with regularization, preconditioning, [11], and conventional GMRES solver has been implemented.

VI. NUMERICAL RESULTS

The outlined theory has been implemented for the first-order Whitney elements and tested on some numerical examples. The aim of the first example is to show the typical shortcomings of 3D node-based finite elements. For this reason we consider a plate of constant magnetic permeability \( \mu_r = 100 \) immersed in the field of a ring-shaped coil which is centered on the long axis of the plate. Figure 2 shows the contour plot of the modulus of the magnetic induction which has been computed using a node-based BE-FE coupling procedure. The jumping field components on the long edges of the plate shows an
insufficient approximation by nodal elements. This lack of the required discretization properties can be remedied completely neither by the mesh refinement nor by the increasing of the finite elements order.

The edge-based BE-FE coupling algorithm achieves an adequate approximation of the physically correct solution which is depicted in Figure 3.

Next, we consider a 3D model of the magnetic valve shown in Figure 4. The importance of the 3D effects of the magnetic flux requires a three-dimensional modelling of the valve. The non-linear magnetostatic problem has been solved by the edge-based BE-FE coupling procedure described above with approximately 60,000 degrees of freedom. Due to the smoothness the correct solution could be obtained by applying the node-based solver, too, using second-order nodal elements with approximately 210,000 unknowns. The results of the edge-based solver are presented in Figure 4 and show a very good agreement with those of the node-based computation.

VII. CONCLUSIONS

In this paper we presented the BE and FE formulations for Maxwell’s equations in terms of differential forms. The discretization of the electromagnetic fields has been accomplished by Whitney forms and the novel BE collocation formulation by DeRham maps generalizing the classical point collocation method has been presented. The coupling procedure of the BE and FE methods has been described and the algebraic properties of the resulting system matrix as well as the corresponding solver strategies have been addressed. The considered numerical examples have demonstrated the feasibility of the edge-based solver in application to the problems with non-smooth solutions as well as its applicability to industrially relevant problems.

REFERENCES