Multi-bunch instability with uneven fills

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Outline of the talk

- Motivation
- Standard theory of the multi-bunch instability (MBI) with a uniform fill
- RW MBI in FCC-ee
- Description of MBI as 1D medium oscillations for non-uniform fills
- Examples of MBI in uneven fills for RW wake
- Summary
Multi-bunch transverse instability with a uniform fill, $M$ bunches. Treat bunches as point charges with charge equal to $N e$:

\[
\ddot{y}_n(t) + \omega_B^2 y_n(t) = -\frac{N r_0 c}{\gamma T_0} \sum_k \sum_{m=0}^{M-1} W_1 \left( -k C - \frac{m-n}{M} C \right) \times y_m \left( t - k T_0 - \frac{m-n}{M} T_0 \right). \tag{4.110}
\]

Assume time dependence

\[
y_n(t) = \tilde{y}_n e^{-i\omega t}
\]

Guess the mode structure ($0 \leq l \leq M - 1$)

\[
\tilde{y}_n^{(l)} \propto e^{2\pi iln/M}
\]

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Multi-bunch instability

Frequency shift of mode \( l \), \( \Delta \omega(l) = \omega(l) - \omega_\beta \), assuming \( |\Delta \omega(l)| \ll \omega_\beta \):

\[
\Delta \omega(l) = -i \frac{M N r_e c}{2 \gamma T_0^2 \omega_\beta} \sum_{p=-\infty}^{\infty} Z_t[\omega_\beta + (pM + l) \omega_0]
\]

where \( Z_t(\omega) \) is the transverse impedance for the whole ring, \( M \) is the number of bunches in the ring, \( N \) is the number of particles in the bunch, \( r_e \) is the electron classical radius, \( \gamma \) is the relativistic factor, \( T_0 = C/c \) is the revolution period in the ring, \( C \) is the circumference, \( l \) is an integer number of the mode, and \( \omega_\beta \) is the betatron frequency.

For the resistive wall the transverse wake is

\[
\nu_t(z) = \frac{2C}{\pi b^3} \sqrt{\frac{c}{\sigma}} \frac{1}{z^{1/2}}, \quad Z_t(\omega) = \sqrt{\frac{2}{\pi}} \frac{\text{sgn}(\omega) - i}{b^3 \sqrt{\sigma|\omega|}}
\]

where \( b \) is the pipe radius, \( \sigma \) is the wall conductivity (assuming the skin depth small and bunches not very short).
The sum over $p$ can be computed analytically in terms of the polylogarithm function $\text{Li}_{\frac{1}{2}}(x)$: $\text{Li}_k(x) = \sum_{n=1}^{\infty} (x^n / n^k)$,

$$\Delta \omega(l) = -\frac{N r_{e c}}{2 \gamma T_0 \omega_\beta} w_t(s_b) \text{Li}_{\frac{1}{2}}(e^{2\pi i(l+\nu_\beta)/M})$$

where $s_b = C/M$.

The maximum growth rate is attained for the minimal value of the argument $|l + \nu_\beta|$, when $l + \nu_\beta < 0$. This occurs for $l = -\{\nu_\beta\} - 1$ with $-(1 - \lfloor \nu_\beta \rfloor)$, where $\{\nu_\beta\}$ is the integer and $\lfloor \nu_\beta \rfloor$—the fractional part of the tune. For $-x \ll 1$ ($M \gg 1$), we have $\text{Li}_{\frac{1}{2}}(e^{2\pi ix}) \approx (1-i)/2\sqrt{-x}$ which gives

$$\text{Im} \Delta \omega_{\text{max}} = \frac{N r_{e c}}{4 \gamma T_0 \omega_\beta} w_t(s_b) \sqrt{\frac{M}{1 - \lfloor \nu_\beta \rfloor}}$$
Application to FCC-ee

\[ C = 100 \text{ km}, \ E = 45 \text{ GeV}, \ I = 1.45 \text{ A}, \ M = 90000, \ b = 35 \text{ mm}, \]
\[ \text{aluminum vacuum chamber}, \ \nu^{}_{\beta} = 350.05 \]
\[ \text{Im} \Delta \omega_{\text{max}} = 460 \text{ s}^{-1} \]

The growth time is about 7 turns.

With 400 MHz RF frequency the collider has \( \approx 130000 \) buckets, so 90000 will not be filled uniformly (in another option \( M = 30000 \)).

How does a non-even fill affect the growth rate of the instability? The original formulation from Chao’s book can be generalized for arbitrary distances between the bunches and then solved numerically. However, for 90000 bunches this may be problematic. We would like to have a simpler description of the problem.

See the animations in Mathematica.
The most unstable modes have $|l| \approx \nu \beta$ with the wavelength $\sim 2\pi c/\omega \beta$ of the mode, that is typically much larger than the distance between the bunches. This means that we can use a continuous approximation for the beam to study this instability.

The beam is treated as a continuous medium with an offset $y(s, t)$. We will seek solutions in the form

$$y(s, t) = y_0(s - ct)e^{-i\omega t}$$

where $y_0$ is the amplitude that depends on the location in the beam. For a given bunch in the train $s - ct = \text{const}$, hence $\omega$ is the frequency of oscillations for a given bunch. Function $y_0$ is a periodic function of its argument $s - ct$ with the period $C$. 
I also assume the beam current

\[ I(s, t) = I_0(s - ct) \]

(for a uniform distribution \( I(s, t) = \text{const} \)). Function \( I_0 \) is a periodic function of its argument \( s - ct \) with the period \( C \).

An example of a fill with a relative gap \( w \): function \( I_0(s - ct) \).
Force acting on the beam

We assume that the wake does not propagate and stays at the location where it was generated (a cavity wake, rw wake, etc.). The force acting per unit charge located at \( s \) at time \( t \) is generated by an infinitesimal part of the beam \( ds' \) that was located at \( s \) at an earlier time \( t' = t - (s' - s)/c \), with \( s' > s \) (\( s \) is measured in the direction of motion, so \( s' > s \) means particles ahead). This force is proportional to the offset at that time at \( s \), \( y(s, t - (s' - s)/c) \) and the current at that location at that time \( I(s, t - (s' - s)/c) = I_0(s' - ct) \). The force is

\[
dF_y(s, t) = \frac{ds'}{c} I_0(s' - ct) y\left(s, t - \frac{s' - s}{c}\right) \omega_t(s' - s)
\]

\[
= \frac{ds'}{c} I_0(s' - ct) y_0(s' - ct) \omega_t(s' - s) \exp \left[-i\omega \left(t - \frac{s' - s}{c}\right)\right]
\]
Equation of motion of the fluid

To find the total force we integrate over \( s' \) from \( s \) to infinity. Integration from \( s' = s \) to \( s' = s + C \), where \( C \) is the ring circumference corresponds to the integration over the particles ahead of the given one. Integration over the interval \( s' > s + C \) means taking into account contributions to the wake from the previous turns. In this integration we need to keep in mind that \( I_0 \) and \( y_0 \) are periodic functions of their arguments, however \( \omega_t \), of course, is not periodic, as well as the exponential function \( \exp[-i\omega (t - (s' - s)/c)] \). With \( z = s' - s \) we have

\[
F_y(s, t) = \frac{1}{c} \int_0^\infty dz I_0(z + s - ct)y_0(z + s - ct)\omega_t(z) \exp \left[-i\omega \left(t - \frac{z}{c}\right)\right]
\]

We now substitute this into equation of motion

\[
\frac{d^2y}{dt^2} + \omega^2 y = \frac{eF_y}{m\gamma}
\]

(note that now \( d/dt = \partial/\partial t + c\partial/\partial s \)).
Integral equation for eigenmodes

Introducing the variable $\xi = s - ct$ we arrive at the following linear integral equation

\[(\omega^2 - \omega^2_{\beta})y_0(\xi) = -\frac{e}{mc\gamma} \int_0^\infty dz I_0(z + \xi)y_0(z + \xi)w_t(z)e^{i\omega z/c}\]

Let us reproduce the result for a uniform fill assuming that $I_0$ is constant. We then have

\[(\omega^2 - \omega^2_{\beta})y_0(\xi) = -\frac{eI_0}{mc\gamma} \int_0^\infty dz y_0(z + \xi)w_t(z)e^{i\omega z/c}\]

Because $y_0$ is periodic we seek an eigen-solution in the for $y_0(\xi) \propto e^{ik_z\xi}$ with $k_z = 2\pi l/C$ with integer $l$. This gives

\[(\omega^2 - \omega^2_{\beta}) = -\frac{ieI_0}{mc\gamma} \int_0^\infty dz w_t(z)e^{iz(k_z + \omega/c)} = -\frac{ieI_0}{m\gamma} Z_t(\omega + ck_z)\]
Reproducing result for uniform fill

Assuming $|\Delta \omega(k_z)| \ll \omega_\beta$

$$\Delta \omega(k_z) = -\frac{ieI_0}{2\omega_\beta m\gamma} Z_t(ck_z + \omega_\beta)$$

Because $\text{Re } Z_t(\omega) \propto \text{sgn}(\omega)|\omega|^{-1/2}$ the most unstable mode has a minimal absolute value of $ck_z + \omega_\beta < 0$. This result is in full agreement with the analysis that considers bunches as point charges for the modes that involve many bunches.

The modes propagating in the direction of the beam are unstable, and the most unstable ones are with the minimal frequency $|ck_z + \omega_\beta|$ at fixed $s$. 


Arbitrary fill pattern

Go back to the main equation. Using the periodicity of \( I_0 \) and \( y_0 \) we expand them into Fourier series

\[
I_0(\xi) = \tilde{I} \sum_{p=-\infty}^{\infty} d_p e^{ipq\xi}, \quad y_0(\xi) = \sum_{n=-\infty}^{\infty} y_n e^{inq\xi},
\]

where \( q = 2\pi/C \) and \( \tilde{I} \) is the average current in the ring \( (d_0 = 1) \).

We now use the resistive wall impedance and normalize the frequency by the maximal growth rate of a uniform fill in the limit of the tune close to an integer \( (\nu_\beta = \text{integer} + \epsilon) \)

\[
\delta \omega = \Omega \frac{e\tilde{I}}{\sqrt{2\pi m \gamma \omega_\beta b^3 \sqrt{\sigma \omega_0}}},
\]

\[
\Delta \omega y_n = -\frac{ie\tilde{I}}{2\omega_\beta m \gamma} Z_t(\omega_0(\nu_\beta + n)) \sum_{m=-\infty}^{\infty} y_m d_{n-m}
\]
We will also change the indexing using $n = p - \{\nu_{\beta}\} - 1$ and $m = p' - \{\nu_{\beta}\} - 1$. For the uniform fill, the most unstable mode has $p = 0$ and the most stable one is $p = 1$. We obtain

$$\Omega y_p = -\frac{1 + i \text{sgn}(\nu_{\beta} + p - 1)}{\sqrt{|\nu_{\beta} + p - 1|}} \sum_{l'=-\infty}^{\infty} y_{p'} d_{p-p'}$$

For a given current distribution in the ring we calculate the coefficients $d_n$, truncate the infinite system of linear equations and find its eigenvalues and eigenvectors. The imaginary parts of the eigenvalues gives the growth rates of the eigenmodes.
Gap in the bunch train

\[ I_0 \]

\[ 0 \]

\[ 0 \]

\[ s-ct \]

\[ wC \]

\[ C \]
Uniform fill, $\nu = 0$

$[\nu_\beta] = 0.05$, the system is truncated to 41 equation

```
p=0 unstable mode
p=1 stable mode
```

```
\Omega = \Omega_0 \Omega_1 \Omega_2 \Omega_3 \Omega_4
n = n_0 n_1 n_2 n_3 n_4
```

```
\nu = 0
```

```
\nu_\beta = 0.05
```

```
\nu_\beta \Omega_0 \Omega_1 \Omega_2 \Omega_3 \Omega_4
n = n_0 n_1 n_2 n_3 n_4
```
Uniform fill, $\nu = 0$, two modes

The harmonics contents of two modes

![Graphs showing the harmonics contents of two modes](image)
Gap $w = 0.3$ in the bunch train

$[\nu_\beta] = 0.05$, gap $w = 0.3$ with the same beam current $\bar{I}$ as in a uniform fill.

Red—uniform fill, blue—a fill with 0.3 gap, 41 modes, green—a fill with 0.3 gap, 61 modes.
Gap $\omega = 0.3$ in the bunch train
Summary

- With the number of bunches in the collider $M \gg 1$ one can use a continuous approximation to calculate the transverse MBI with non-uniform fills. An integral equation is derived for the coherent frequency shift of the beam oscillations in fluid approximation. For a given fill pattern, this equation can be easily solved using the Fourier expansion.

- For the RW MBI, gaps up to $w = 0.3$ do not noticeably change the growth rate of the most unstable mode (assuming the same averaged current of the beam).

I am interested in finding other applications for this technique and making comparison with computer simulations.