DELPHI: an analytic Vlasov solver for impedance-driven modes

N. Mounet

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Outline

- Introduction & motivation
- Getting to Sacherer integral equation
- How to solve Sacherer equation: Discrete Expansion over Laguerre Polynomials and Headtail modes
- Some benchmarks
- Landau damping
- What else could be done?
• For given machine and beam parameters, we often need to evaluate transverse beam stability w.r.t. impedance effects.
• Also important to assess the efficiency of stabilization techniques: damper, non-linearities (Landau damping).
• Time domain macroparticles tracking is often a very good tool and can give a complete vision, BUT:
 ➢ Too slow for certain large scale problems, e.g. typical LHC problem (~1400/2800 bunches with transverse damper → need fine modeling of intrabunch motion and more than 100000 turns to see an instability...),
 ➢ Too slow to perform large parameter space scans (chromaticity, non-linearities, intensity, damper gain, etc.),
 ➢ Very difficult to be sure the beam is stable in a certain configuration.
Introduction & motivation

- Another possibility: Vlasov solver in "mode domain"
  → solver that looks for all modes that can develop, among which one can easily spot the most critical (i.e. unstable).

- Idea is not new (non exhaustive list):
  - Laclare formalism [J. L. Laclare, CERN-87-03-V-1, p. 264],
  - MOSES [Y. Chin, CERN/SPS/85-2 & CERN/LEP-TH/88-05],
  - NHTVS [A. Burov, Phys. Rev. ST AB 17, 021007 (2014)].

  → all linear collective dynamics represented by a matrix; then eigenvalues = modes.
All current Vlasov solvers have limitations:

- **Laclare** cannot solve problems that involve too many betatron sidebands (give size of matrix to diagonalize),
- **MOSES** limited to single-bunch, resonator models, w/o damper,
- **NHTVS** does not automatically check convergence, relies on airbag rings for radial discretization & treats Landau damping in the framework of stability diagram theory (approximation).

→ Can we do better?
Getting to Sacherer integral equation

- Outline:
  - Vlasov equation
  - Hamiltonian
  - Perturbative approach adopted
  - Impedance term

→ Sacherer integral equation for transverse modes.


- Note: here, unlike Chao we use "engineer" convention for the Fourier transform → $e^{j\omega t}$ (unstable modes have $\text{imag. part}<0$). Also, SI units (c.g.s in Chao), and *notations* often different.
Vlasov equation
[A. A. Vlasov, J. Phys. USSR 9, 25 (1945)]

- Vlasov equation expresses that the local phase space density does not change when one follows the flow of particles.
- In other words: local phase space area is conserved in time:

\[
\text{Assumptions:}
\]

- conservative & deterministic system (governed by Hamiltonian) – no damping or diffusion from external sources (no synchrotron radiation),
- external forces (no discrete internal force or collision).

→ impedance seen as a collective field from ensemble of particles.

Figure 6.3. (a) Phase space distribution of particles at time \( t \). A rectangular box \( ABCD \) with area \( \Delta q \Delta p \) is drawn and magnified. (b) At a later time, \( t + dt \), the box moves and deforms into a parallelogram with the same area as \( ABCD \). All particles inside the box move with the box.

Courtesy A. W. Chao
Simplest expression: with $\psi$ the general 6D phase space distribution density (and $t$ the time),

$$\frac{d\psi}{dt} = 0,$$

In our case:

- independent variable chosen as $s = v \times t$ (longitudinal position along accelerator orbit),
- particle coordinates (4D – no x/y coupling):
  - transverse: $(y, p_y) \Leftrightarrow (J_y, \theta_y)$ (action/angle)
  - longitudinal: $(z, \delta)$

$$\frac{\partial\psi}{\partial s} + J'_y \frac{\partial\psi}{\partial J_y} + \theta'_y \frac{\partial\psi}{\partial \theta_y} + z' \frac{\partial\psi}{\partial z} + \delta' \frac{\partial\psi}{\partial \delta} = 0.$$
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$$\frac{\partial \psi}{\partial s} + J'_y \frac{\partial \psi}{\partial J_y} + \theta'_y \frac{\partial \psi}{\partial \theta_y} + z' \frac{\partial \psi}{\partial z} + \delta' \frac{\partial \psi}{\partial \delta} = 0.$$
In the presence of a dipolar vertical impedance resulting in a force $F_y(z,s)$:

\[
H = \frac{Q_y}{R} J_y - \frac{1}{2\eta} \left( \frac{\omega_s}{u} \right)^2 z^2 - \frac{\eta}{2} \delta^2 - \frac{y}{E} F_y(z, s)
\]

with

\[
Q_y = Q_{y0} + Q'_y \delta,
\]

and

\[
J_y = \frac{1}{2} \left( \frac{Q_{y0}}{R} y^2 + \frac{R}{Q_{y0}} p_y^2 \right),
\]

\[
y = \sqrt{2 J_y \frac{R}{Q_{y0}}} \cos \theta_y,
\]

\[
p_y = \sqrt{2 J_y \frac{Q_{y0}}{R}} \sin \theta_y.
\]
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with

\[ Q_y = Q_{y0} + Q_y' \delta, \]

and

\[ J_y = \frac{1}{2} \left( \frac{Q_{y0}}{R} y^2 + \frac{R}{Q_{y0}} p_y^2 \right), \]

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and

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$$y = \sqrt{\frac{2J_y R}{Q_{y0}}} \cos \theta_y,$$

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\[ H = \frac{Q_y}{R} J_y - \frac{1}{2\eta} \left( \frac{\omega_s}{v} \right)^2 z^2 - \frac{\eta}{2} \delta^2 - \frac{y}{E} F_y(z, s) \]

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\[ J_y = \frac{1}{2} \left( \frac{Q_{y0}}{R} y^2 + \frac{R}{Q_{y0}} p_y^2 \right), \]

\[ y = \sqrt{2J_y} \frac{R}{Q_{y0}} \cos \theta_y, \]

\[ p_y = \sqrt{2J_y} \frac{Q_{y0}}{R} \sin \theta_y. \]

→ important assumption: invariant (and action-angle variables) stay as in linear case...
Hamilton's equations...

- ... will then give derivatives of $J_y$, $\theta_y$, $z$ and $\delta$ w.r.t. $s$:

\[
J'_y = -\frac{\partial H}{\partial \theta_y} = \frac{\partial y}{\partial \theta_y} \frac{F_y(z, s)}{E},
\]

\[
\theta'_y = \frac{\partial H}{\partial J_y} = \frac{Q_y}{R} - \frac{\partial y}{\partial J_y} \frac{F_y(z, s)}{E},
\]

\[
z' = \frac{\partial H}{\partial \delta} = -\eta \delta + \frac{Q'_y}{R} J_y,
\]

\[
\delta' = \frac{\partial H}{\partial z} = \left(\frac{\omega_s}{\nu}\right)^2 z + \frac{y}{E} \frac{\partial F_y}{\partial z}.
\]
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$$
J'_y = -\frac{\partial H}{\partial \theta_y} = \frac{\partial y}{\partial \theta_y} \frac{F_y(z, s)}{E},
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$$

$$
z' = \frac{\partial H}{\partial \delta} = -\eta \delta + \frac{Q'_y}{R} J_y,
$$

$$
\delta' = -\frac{\partial H}{\partial z} = \left(\frac{\omega_s}{v}\right)^2 \frac{z}{\eta} + \frac{y}{E} \frac{\partial F_y}{\partial z}.
$$

**Neglected** (not even mentioned in Chao's book)

**Neglected** (from Chao: OK when far from synchro-betatron resonances & small transverse beam size)
How to solve Vlasov equation?

- Equation remains quite complicated: partial differential eq. for distribution function $\psi (s, J_y, \theta_y, z, \delta)$:

\[
\frac{\partial \psi}{\partial s} + J'_y \frac{\partial \psi}{\partial J_y} + \theta'_y \frac{\partial \psi}{\partial \theta_y} + z' \frac{\partial \psi}{\partial z} + \delta' \frac{\partial \psi}{\partial \delta} = 0.
\]

- To simplify the problem:
  
  ➢ Assume a mode is developing in the bunch along the revolutions, with a certain (complex) frequency $\Omega = Q c \omega_0$,
  
  ➢ Assume we stay close to the stationary unperturbed distribution $\psi_0$, function of invariants $J_y$ and → perturbation formalism:

\[
\psi = f_0(J_y) g_0(r) + f_1(J_y, \theta_y) g_1(z, \delta) e^{j \Omega s / \nu}.
\]
Equation remains quite complicated: partial differential eq. for distribution function $\psi(s, J_y, \theta_y, z, \delta)$:

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→ perturbation formalism:

$$\psi = f_0(J_y)g_0(r) + f_1(J_y, \theta_y)g_1(z, \delta) e^{\frac{j \Omega s}{\omega_s}}.$$
Use polar coordinates in longitudinal:

\[
\begin{align*}
z &= r \cos \phi, \\
\eta v \delta &= r \sin \phi.
\end{align*}
\]

After some algebra, neglecting second order terms proportional to $\Delta \psi_1 F_y$ (wake field force assumed to be small):

\[
e^{\frac{j \Omega_s}{v}} \left( f_1 g_1 \frac{j \Omega_s}{v} + \frac{Q_y}{R} \frac{\partial f_1}{\partial \theta_y} g_1 + \frac{\omega_s}{v} f_1 \frac{\partial g_1}{\partial \phi} \right) =
\]

\[
\frac{\sin \theta_y}{E} \sqrt{\frac{2J_y R}{Q_y_0}} F_y(z, s) g_0(r) f'_0(J_y).
\]
Rewriting Vlasov equation

- Use polar coordinates in longitudinal:

\[
\begin{align*}
z &= r \cos \phi, \\
\frac{\eta \nu}{\omega_s} \delta &= r \sin \phi.
\end{align*}
\]

- After some algebra, neglecting second order terms proportional to \( \Delta \psi_1 F_y \) (wake field force assumed to be small):

\[
e^{j \Omega s/v} \left( f_1 g_1 \frac{j \Omega s}{v} + \frac{Q_y}{R} \frac{\partial f_1}{\partial \theta_y} g_1 + \frac{\omega_s}{v} f_1 \frac{\partial g_1}{\partial \phi} \right) =
\]

\[
\frac{\sin \theta_y}{E} \sqrt{2J_y R \frac{Q_y 0}{Q_y 0}} F_y (z, s) g_0 (r) f_0' (J_y).
\]

\[
Q_y = Q_{y0} + Q'_y \delta,
\]
The trick is to find appropriate decompositions...

- Writing $f_1$ as a Fourier series

$$f_1(J_y, \theta_y) = \sum_{k=-\infty}^{+\infty} f_1^k(J_y) e^{jk\theta_y},$$

we can show that all $f_1^k$ are zero except for $k=-1$ (this is exact except for $k=1$ for which it relies on $|Q_c - Q_y| \ll |Q_c + Q_y|$)

$$\rightarrow f_1(J_y, \theta_y) = f(J_y) e^{-j\theta_y}.$$

- For $g_1$ decomposition is more subtle:
The trick is to find appropriate decompositions...

- Writing $f_1$ as a Fourier series

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f_1(J_y, \theta_y) = \sum_{k=-\infty}^{+\infty} f_1^k(J_y) e^{jk\theta_y},
\]

we can show that all $f_1^k$ are zero except for $k=-1$ (this is exact except for $k=1$ for which it relies on $|Q_c - Q_y| << |Q_c + Q_y|$)

\[\rightarrow f_1(J_y, \theta_y) = f(J_y) e^{-j\theta_y}.\]

- For $g_1$ decomposition is more subtle:

\[
g_1(r, \phi) = e^{-jQ'y z / \eta R} \sum_{l=-\infty}^{+\infty} R_l(r) e^{-jl\phi}, \rightarrow \text{azimuthal modes decomposition}
\]
The trick is to find appropriate decompositions…

- Writing \( f_1 \) as a Fourier series

\[
f_1(J_y, \theta_y) = \sum_{k=-\infty}^{+\infty} f_1^k(J_y) e^{jk\theta_y},
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we can show that all \( f_1^k \) are zero except for \( k=-1 \) (this is exact except for \( k=1 \) for which it relies on \(|Q_c - Q_y| \ll |Q_c + Q_y| \))

\[
\rightarrow f_1(J_y, \theta_y) = f(J_y) e^{-j\theta_y}.
\]

- For \( g_1 \) decomposition is more subtle:

\[
g_1(r, \phi) = e^{\frac{-jQ'_y}{\eta R}} \sum_{l=-\infty}^{+\infty} R_l(r) e^{-jl\phi}, \quad \rightarrow \text{azimuthal modes decomposition}
\]

to cancel some term in Vlasov eq.
After such decompositions, Vlasov eq. now looks like

\[
\sum_{l=-\infty}^{+\infty} R_l(r) \left[ \frac{f(J_y)(Q_c - Q_{y0} - lQ_s)}{f_0'(J_y)\sqrt{2J_y R/Q_{y0}}} \right] e^{-jl\phi} = \frac{Re^{-j\frac{Q_c s}{R}}}{2E} F_y(z, s) e^{\frac{jQ_{y'} z}{\eta R}}
\]
The trick is to find appropriate decompositions...

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\sum_{l=-\infty}^{+\infty} R_l(r) \left[ f(J_y)\left(\frac{Q_c - Q_{y0} - lQ_s}{f'_0(J_y)\sqrt{2J_yR/Q_{y0}}}\right)\right] e^{-jl\phi} = \frac{Re^{-j\frac{Q_c\phi}{R}}}{2E} F_y(z, s) e^{\frac{jQ'_y z}{nR}}
\]

must be constant w.r.t \( J_y \) → dipole mode
The trick is to find appropriate decompositions...

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\]

must be constant w.r.t $J_y$ → dipole mode

- Next step is to evaluate $F_y$:
  - written initially as a 4D integral (convolution of the wake in $z$, weighted by total distribution function):

\[
F_y(z, s) = \frac{e^2}{2\pi R} \sum_{k=\infty}^{+\infty} e^{jQ_c(s-2\pi k R)R} \int \int \int d\tilde{z} d\tilde{\delta} dJ_y d\theta_y \psi(\tilde{z}, \tilde{\delta}, J_y, \theta_y) y W_y(z - \tilde{z}).
\]
The trick is to find appropriate decompositions...

- After such decompositions, Vlasov eq. now looks like

\[
\sum_{l=-\infty}^{+\infty} R_l(r) \left[ \frac{f(J_y)(Q_c - Q_{y0} - lQ_s)}{f'_0(J_y)\sqrt{2J_y R/Q_{y0}}} \right] e^{-jl\phi} = \frac{Re^{-j\frac{Q_c s}{R}}}{2E} F_y(z, s) e^{\frac{jQ'_y s}{\eta R}}
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\]

multiturn sum (importance of self-consistency)
Using the fact that the unperturbed distribution has no dipole moment, and the previous decompositions:

\[
F_y(z, s) = \frac{e^2}{2Q_y} e^{iQ_c s} \sum_{l=-\infty}^{+\infty} \left( D \int_0^{+\infty} dJ_y f'_0(J_y) J_y \right) \cdot \\
+ \sum_{k=-\infty}^{+\infty} e^{-2\pi i Q_c k} \int d\tilde{z} W_y(z - \tilde{z}) \int d\delta e^{-jQ'_y \tilde{z}} R_l(r) e^{-j l \phi}.
\]

\[
S(z)
\]
How to write the wake fields force

- Using the fact that the unperturbed distribution has no dipole moment, and the previous decompositions:

\[
F_y(z, s) = \frac{e^2}{2Q_y0} e^{jQcs/R} \sum_{l=-\infty}^{+\infty} \left( D \int_0^{+\infty} dJ_y f_0'(J_y) J_y \right) \cdot \sum_{k=-\infty}^{+\infty} e^{-2\pi jQck} \int dzW_y(z - \tilde{z}) \int d\delta e^{\frac{-jQ'\delta}{\eta R}} R_l(r) e^{-jl\phi}.
\]

- After some tricks we get for an **impedance** (details in Chao):

\[
S(z) = -\frac{vQ_s}{\eta R^2} j^{-l} \sum_{p=-\infty}^{+\infty} e^{-j(Q_c + p) \frac{\pi}{8}} Z_y \left[ -\omega_0(Q_c + p) \right] \int_0^{+\infty} r R_l(r) J_l \left[ (\omega_\xi - \omega_0(Q_c + p)) \frac{r}{v} \right] dr.
\]

and for an ideal **damper** (constant imag. wake, no multiturn):

\[
S(z) \propto -\frac{vQ_s}{\eta R^2} j^{-l} \int_0^{+\infty} r R_l(r) J_l \left( \frac{\omega_\xi r}{v} \right) dr.
\]
Using the fact that the unperturbed distribution has no dipole moment, and the previous decompositions:

\[ F_y(z, s) = \frac{e^2}{2Q_y} e^{\frac{jQ_e s}{R}} \sum_{l=-\infty}^{+\infty} \left( D \int_0^{+\infty} dJ_y f_l^0(J_y) J_y \right) \cdot \]

\[ + \sum_{k=-\infty}^{+\infty} e^{-2\pi j Q_c k} \int d\tilde{z} W_y(z - \tilde{z}) \int d\tilde{\delta} e^{-\frac{jQ'_y \tilde{z}}{\eta R}} R_l(r) e^{-jl\phi}. \]

After some tricks we get for an impedance (details in Chao):

\[ S(z) = -\frac{vQ_s}{\eta R^2} j^{-l} \sum_{p=-\infty}^{+\infty} e^{-j(Q_c + p)\tilde{\eta}} Z_y [-\omega_0(Q_c + p)] \int_0^{+\infty} r R_l(r) J_l \left[ (\omega - \omega_0(Q_c + p)) \frac{r}{v} \right] dr. \]

and for an ideal damper (constant imag. wake, no multiturn):

\[ S(z) \propto -\frac{vQ_s}{\eta R^2} j^{-l} \int_0^{+\infty} r R_l(r) J_l \left( \frac{\omega r}{v} \right) dr. \]
Finally, combining the 2 previous slides, integrating over $\phi$, defining $\tau = r/v$ (maximum long. amplitude in seconds), neglecting $Q_c - Q_y$ in the impedance and Bessel functions, and generalizing to $M$ equidistant bunches of intensity per bunch $N$ with the usual assumption they all oscillate in the same way:

\[
(\Omega - Q_y \omega_0 - l \omega_s) R_l(\tau) =
\]

\[
- \kappa g_0(\tau) \sum_{l'=-\infty}^{+\infty} j^{l'-l} \int_0^{+\infty} \tau' R_l(\tau') \left[ \frac{\mu}{\omega_0} J_l(-\omega_\xi \tau) J_{l'}(-\omega_\xi \tau') \right] d\tau' +
\]

\[
+\infty \sum_{p=-\infty}^{+\infty} Z_y(\omega_p) J_l((\omega_\xi - \omega_p)\tau) J_{l'}((\omega_\xi - \omega_p)\tau')
\]

\]

The DELPHI Vlasov solver - N. Mounet - HSC meeting 07/05/2014
Finally, combining the 2 previous slides, integrating over $\phi$, defining $\tau = r/\nu$ (maximum long. amplitude in seconds), neglecting $Q_c - Q_{y0}$ in the impedance and Bessel functions, and generalizing to $M$ equidistant bunches of intensity per bunch $N$ with the usual assumption they all oscillate in the same way:

\begin{equation}
(\Omega - Q_{y0}\omega_0 - l\omega_s)R_l(\tau) = \kappa g_0(\tau) \sum_{l'=-\infty}^{+\infty} j^{l'-l} \int_{0}^{+\infty} \tau' R_l(\tau') \left[ \frac{\mu}{\omega_0} J_l(-\omega_\xi \tau) J_{l'}(-\omega_\xi \tau') \, d\tau' + \sum_{p=-\infty}^{+\infty} Z_y(\omega_p) J_l((\omega_\xi - \omega_p)\tau) J_{l'}((\omega_\xi - \omega_p)\tau') \right].
\end{equation}

\[
\kappa = -j \frac{N f_0 e^2 M}{2\gamma m_0 c Q_{y0}}, \quad \frac{-\kappa \mu}{\omega_0} = j \frac{f_0}{n_d},
\]

\[
\omega_p = (n + pM + [Q_{y0}])\omega_0,
\]

n° damping turns

coupled-bunch mode
tune fractional part
Finally, combining the 2 previous slides, integrating over $\phi$, defining $\tau = r/v$ (maximum long. amplitude in seconds), neglecting $Q_c - Q_{y0}$ in the impedance and Bessel functions, and generalizing to $M$ equidistant bunches of intensity per bunch $N$ with the usual assumption they all oscillate in the same way:

\[
(\Omega - Q_{y0}\omega_0 - l\omega_s)R_l(\tau) = -\kappa g_0(\tau) \sum_{l'=-\infty}^{+\infty} j^{l'} - l \int_0^{+\infty} \tau' R_l(\tau') \left[ \frac{\mu}{\omega_0} J_l(-\omega_x \tau) J_{l'}(-\omega_x \tau') \, d\tau' \right] + \sum_{p=-\infty}^{+\infty} Z_y(\omega_p) J_l((\omega_x - \omega_p)\tau) J_{l'}((\omega_x - \omega_p)\tau') \right].
\]

\[
\kappa = -j \frac{N f_0 e^2 M}{2\gamma m_0 c Q_{y0}}, \quad \frac{-\kappa \mu}{\omega_0} = j \frac{f_0}{nd}, \quad \omega_p = (n + pM + \lfloor Q_{y0} \rfloor)\omega_0.
\]
Finally, combining the 2 previous slides, integrating over $\phi$, defining $\tau = r/v$ (maximum long. amplitude in seconds), neglecting $Q_c - Q_{y0}$ in the impedance and Bessel functions, and generalizing to $M$ equidistant bunches of intensity per bunch $N$ with the usual assumption they all oscillate in the same way:

\[
(\Omega - Q_{y0}\omega_0 - l\omega_s) R_l(\tau) = \kappa g_0(\tau) \sum_{l'=-\infty}^{+\infty} j^{l'-l} \int_0^{+\infty} \tau' R_l(\tau') \left[ \frac{\mu}{\omega_0} J_l(-\omega_\xi \tau) J_{l'}(-\omega_\xi \tau') d\tau' + \sum_{p=-\infty}^{+\infty} Z_y(\omega_p) J_l((\omega_\xi - \omega_p)\tau) J_{l'}((\omega_\xi - \omega_p)\tau') \right].
\]

$\kappa = -j \frac{N f_0 e^2 M}{2\gamma m_0 c Q_{y0}}$,

$\frac{-\kappa \mu}{\omega_0} = \frac{j f_0}{n d}$,

$\omega_p = \left( n + pM + \left[Q_{y0}\right] \right) \omega_0$.
How are we going to solve this?

- Using a decomposition over Laguerre polynomials of the radial functions (idea from Besnier 1974, used then by Y. Chin in code MOSES – 1985):

\[
R_l(\tau) = \left( \frac{\tau}{\tau_b} \right)^{|l|} e^{-b\tau^2} \sum_{n=0}^{+\infty} c_n^l L_n^{|l|}(a\tau^2),
\]

\[
g_0(\tau) = e^{-b\tau^2} \sum_{k=0}^{n_0} g_k L_k^0(a\tau^2),
\]

→ in principle any long distribution can be put in.
How are we going to solve this?

- Using a decomposition over Laguerre polynomials of the radial functions (idea from Besnier 1974, used then by Y. Chin in code MOSES – 1985):

\[ R_l(\tau) = \left( \frac{\tau}{\tau_b} \right)^{|l|} e^{-b\tau^2} \sum_{n=0}^{+\infty} c_n^l L_n^{|l|}(a\tau^2), \]

- Then the following integrals can be computed analytically:

\[ \int_{0}^{+\infty} \tau^{1+|l|} L_n^{|l|}(a\tau^2) e^{(b-a)\tau^2} g_0(\tau) J_l(\omega\tau) d\tau, \]

\[ \int_{0}^{+\infty} \tau^{1+|l|} L_n^{|l|}(a\tau^2) e^{-b\tau^2} J_l(\omega\tau) d\tau, \]

\[ \rightarrow \text{in principle any long. distribution can be put in.} \]

\[ \rightarrow \text{can also play with parameters } a \text{ & } b \]

\[ \rightarrow \text{exponentials make impedance sum convergence easy.} \]
In the end Sacherer integral equation can be set as an eigenvalue problem:

\[
(\Omega - Qy_0\omega_0)\alpha_{ln} = \sum_{l'=-\infty}^{+\infty} \sum_{n'=0}^{+\infty} \alpha_{l'n'}(\delta_{ll'}\delta_{nn'}\omega_s + M_{ln,l'n'}). 
\]

In DELPHI, convergence is automatically checked with respect to the matrix size \( (n^r) \text{ radial & azimuthal modes} \) for every single calculation.

Matrix can be computed only once for a full set of intensities, damper gain or phase, and/or Qs → scans with such parameters can be done quite fast.
Benchmarks

- DELPHI vs MOSES, for single-bunch **TMCI without damper** (LEP RF cavities modelled as a broadband resonator):

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Real part, Q'=0
```

The DELPHI Vlasov solver...
Benchmarks

- DELPHI vs MOSES, for single-bunch TMCI without damper (LEP RF cavities modelled as a broadband resonator):

![Graph showing Imag. part, Q'=0 for LEP, Q'=0, f=0]
Benchmarks

- DELPHI vs MOSES, single-bunch **without damper** (LEP RF cavities modeled as a broadband resonator):

Imag. part, $Q'=22$
Benchmarks

- DELPHI vs Karliner-Popov, single-bunch with damper (VEPP-4, broadband resonator):

\[ \text{VEPP, } Q' = 0.0, f = 2.5, 2 \text{ radial modes, 5 azimuthal modes} \]

Real, part, 
\( Q' = 0 \)
Benchmarks

- DELPHI vs Karliner-Popov, single-bunch with damper (VEPP-4, broadband resonator):

![Graph showing Imag. part, Q'=0]

VEPP, $Q' = 0.0$, $f=2.5$, 2 radial modes, 5 azimuthal modes

Imag. part, $Q' = 0$
**Benchmarks**

- **DELPHI vs Karliner-Popov and HEADTAIL** (macroparticle simulation code – G. Rumolo et al), single-bunch **with damper** (VEPP-4, broadband resonator):

  Imag, part, $Q'=-7.5$

  DELPHI is closer to HEADTAIL.

  Karliner-Popov is more stable → due to their non flat damper? (we cannot check because Karliner-P damper parameters are not provided).
Transverse feedback:

- First idea: reactive feedback (prevent mode 0 to shift down and couple with mode -1) → not more than 5-10 % increase in threshold, despite several attempts and models developed [Danilov-Perevedentsev 1993, Sabbi 1996, Brandt et al 1995],

- Another idea: resistive feedback, first found ineffective [Ruth 1983], tried at LEP but never used in operation. Recently (2005) thought to be a good option by Karliner-Popov with a possible increase by factor ~5 of TMCI threshold → can we confirm?

Impedance model: two broad-band resonators (RF cavities + bellows), the rest is relatively small (<10%) [G. Sabbi, 1995].

→ experimental tune shifts and TMCI threshold (from simple formula) well reproduced,

→ threshold slightly less than 1mA.
- **LEP without damper** (typical LEP2 parameters)

Note: we had to change the bunch length (1.3cm instead of 1.8cm) to match Karliner-Popov's result.
- LEP with resistive damper (typical LEP2 parameters)

Again, we see that Karliner-Popov model gives more stability than DELPHI → we cannot reproduce their result.
LEP: stability analysis with resistive damper

- Instability threshold vs. $Q'$ and damper gain (up to 10 turns) with DELPHI:

Essentially, one cannot do better than the natural (i.e. without damper) TMCI threshold.
LEP: stability analysis with reactive damper

- Instability threshold vs. \( Q' \) and damper gain (up to 10 turns) with DELPHI:

We can do a little better than the "natural" TMCI.

→ seems to match (qualitatively) LEP observations.
To include Landau damping, we simply need to replace the tune by

$$Q_y = Q_{y0} + Q'_y \delta + \frac{a_{yy}}{2} J_y + a_{yx} J_x,$$

Then, assuming the transverse invariant stays ~ the same, the transverse part of the perturbation becomes:

$$f_l(J_x, J_y) = D_l \frac{\frac{\partial f_0(J_x, J_y)}{\partial J_y}}{Q_c - Q_{y0} - a_{yy} J_y - a_{yx} J_x - lQ_s} \sqrt{\frac{2J_y R}{Q_{y0}}};$$

and the expression of the force from dipolar wake fields

$$F_y(z, s) = \frac{e^2}{2Q_{y0}} \sum_{l=\pm \infty} e^{\frac{iQ_c z}{R}} \left( D_l \int \int_{0}^{+\infty} dJ_x dJ_y \frac{\frac{\partial f_0(J_x, J_y)}{\partial J_y} J_y}{Q_c - Q_{y0} - a_{yy} J_y - a_{yx} J_x - lQ_s} \right).$$
We define the dispersion integral as

\[ I_l(Q_c) = \int\int_0^{+\infty} dJ_x dJ_y \frac{\partial f_0(J_x, J_y)}{\partial J_y} J_y \frac{Q_c - Q_{y0} - a_{yy} J_y - a_{yx} J_x - l Q_s}{J_y}, \]

which can be computed analytically for many transverse distributions (Gaussian, parabolic, and others).

Then the equation becomes:

\[ det \left( \left[ \delta_{ll'} \delta_{nn'} \frac{\omega_0}{I_l(Q_c)} \right] + [M_{ln, l'n'}] \right) = 0. \]

This is a non-linear equation of the coherent (complex) tune shift \( Q_c \), which can be solved numerically.
Landau damping with DELPHI: preliminary examples

- With the (old) LHC 2012 model, 5 azimuthal modes and 5 radial modes (non converged computation), **no damper**, single-bunch, \( N_b = 3 \times 10^{11} \) p+/b: comparison with stability diagram theory

→ quite good agreement!
Landau damping with DELPHI: preliminary examples

- With the (old) LHC 2012 model, 5 azimuthal modes and 5 radial modes (non converged computation), 50 turns damper, single-bunch, \( N_b = 3 \times 10^{11} \) p+/b: comparison with stability diagram theory

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The DELPHI Vlasov solver - N. Mounet - HSC meeting 07/05/2014
Conclusions and future possible work

- Developed a new code (DELPHI) to study stability in mode-coupling conditions, with transverse damper, in multibunch, for any longitudinal distribution, and including transverse Landau damping (beyond stability diagram approximation). Benchmarks done (vs. MOSES, Karliner & Popov, HEADTAIL), many more to be done.

- As an example, LEP experimental results (relative ineffectiveness of transverse flat damper – being reactive or resistive) qualitatively obtained.

- In the future, all kinds of longitudinal non-linearities could be included, but with probably some difficulties:
  - non-linear bucket,
  - quadrupolar wakes,
  - second order chromaticity.